

The Entropy of Taub-Bolt Solution

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Abstract. A geometrical framework for the definition of entropy in General Relativity via Nöther theorem is briefly recalled and the entropy of Taub-Bolt Euclidean solutions of Einstein equations is then obtained as an application. The computed entropy agrees with previously known results, obtained by statistical methods. It was generally believed that the entropy of Taub-Bolt solution could not be computed via Nöther theorem, due to the particular structure of singularities of this solution. We show here that this is not true. The Misner string singularity is, in fact, considered and its contribution to the entropy is analyzed. As a result, in our framework entropy does not obey the “one-quarter area law” and it is not directly related to horizons, as sometimes erroneously suggested in current literature on the subject.

1. Introduction

The Taub-Bolt and the Taub-NUT metrics are families of asymptotically locally flat (ALF) solutions of Einstein field equations (without cosmological constant) in Euclidean (positive) signature. We decided to deal with the Euclidean sector because in this case the entropy was already known in literature by the Euclidean path integral method and the Hamiltonian $(3+1)$ treatment of the action functional (see [1], [2] and [3]). In this way we know *a priori* which is the result we are expected to produce. However, since our framework applies without particular *a priori* requirements on signature or dimension also the Lorentzian Taub-NUT metric seems to be susceptible of a similar analysis (though we do not investigate it here).

In these solutions the surfaces of constant radius r have, at infinity, the topology of a non trivial bundle $S^3 \rightarrow S^2$ with fiber S^1 (of constant lenght), instead of the *usual* trivial topology $S^1 \times S^2 \rightarrow S^2$ as in asymptotically flat Euclidean solutions. This non-trivial topology causes the first problem when attempting to derive Taub-Bolt entropy, by trying to dimensionally reduce the action functional along the orbit of the vector ∂_τ , i.e. along the $U(1)$ isometry (see [1]). Deeply related to this kind of problem is the fact that, as already noticed in the literature, Taub-Bolt solution contains a Misner string (see [4]),

i.e. a two dimensional surface singularity running along the z -axis. This singularity is not enclosed by a Killing horizon and it contributes to the entropy. For these reasons it was generally believed that the geometrical approach (see [5], [6] and references quoted therein) based on the integration on cross sections of Killing horizons does not apply to this case. In a suitable sense our framework encompasses and generalizes that “older” approach and shows that these conclusions do not hold. This apparent paradox is due to the following facts: first, entropy is not directly related to cross sections of Killing horizons; secondly, there is no need of requiring horizons to be Killing since there is no need of extending them to bifurcate Killing horizons (see [7]).

We shall here introduce Euclidean, asymptotically locally flat Taub-Bolt solutions together with the geometric notation used to compute their entropy. Further details on Taub-Bolt and Taub-NUT solutions may be found in [1], [4], [8] and [9]. The entropy of Taub-Bolt solutions was already computed in [2] and [3], in the framework for conserved quantities due to Brown and York (see [10]). Our framework is instead based on a direct application of Nöther theorem to a covariant first order Lagrangian for General Relativity (see [7], [11]), but it can be shown to be deeply related to Brown and York framework. In a forthcoming paper of ours we shall investigate the relations between these formalisms (see [12]). We shall not enter here in details, but some similarity has to be noticed: first, both methods compute the *relative conserved quantities and entropy with respect to a fixed background*, regarding *absolute quantities* to be basically undefined; second, both prescriptions reduce to ADM quantities (possibly corrected by “Regge-Teitelboim terms”; see [13]) when the latter applies; third, both methods apply to a wider class of situations than the standard ADM formalism for conserved quantities. However, some differences are to be stressed: first, our method is not restricted to Euclidean signature; second, our analysis is completely covariant and basically independent on a spacetime ADM foliation.

2. Notation

We shall hereafter recall standard geometric notation (see [14] and [15]) as well as the geometric framework for conserved quantities and entropy ([7]).

Let $\mathcal{C} = (C, M, \pi, F)$ be the *configuration bundle* of a field theory and (x^μ, y^i) a system of fibered coordinates. Let us denote by $J^k\mathcal{C}$ the k^{th} *order prolongation* of \mathcal{C} , having $(x^\mu, y^i, y_\mu^i, \dots, y_{\mu_1 \dots \mu_k}^i)$ as *natural fibered coordinates*; $y_{\mu_1 \dots \mu_h}^i$ are understood to denote derivatives of local sections and are therefore meant to be symmetric in their lower indices $(\mu_1 \dots \mu_h)$. If $\Xi = \xi^\mu \partial_\mu + \xi^i \partial_i$ is a projectable vector field over \mathcal{C} , one can define its k^{th}

order prolongation $j^k \Xi = \xi^\mu \partial_\mu + \xi^i \partial_i + \xi_\mu^i \partial_i^\mu + \dots + \xi_{\mu_1 \dots \mu_k}^i \partial_i^{\mu_1 \dots \mu_k}$ over $J^k \mathcal{C}$ by setting

$$\begin{cases} \xi_\mu^i = d_\mu \xi^i - y_\nu^i d_\mu \xi^\nu \\ \dots \\ \xi_{\mu_1 \dots \mu_k}^i = d_{\mu_1} \xi_{\mu_2 \dots \mu_k}^i - y_{\nu \mu_2 \dots \mu_k}^i d_{\mu_1} \xi^\nu \end{cases} \quad (2.1)$$

where

$$d_\mu = \frac{\partial}{\partial x^\mu} + y_\mu^i \frac{\partial}{\partial y^i} + y_{\mu\nu}^i \frac{\partial}{\partial y_\nu^i} + \dots + y_{\mu\nu_1 \dots \nu_h}^i \frac{\partial}{\partial y_{\nu_1 \dots \nu_h}^i} + \dots \quad (2.2)$$

denotes the *total derivative operator* (at any order).

The bundle \mathcal{C} is assumed to be a *natural bundle* (see [15]): then there exists an action of spacetime diffeomorphisms $\text{Diff}(M)$ on \mathcal{C} , i.e. a canonical group homomorphism $\text{Diff}(M) \rightarrow \text{Aut}(\mathcal{C})$ into the automorphisms of \mathcal{C} . Any natural bundle is associated to the s -frame bundle $L^s(M)$ for some finite and minimal order s (see [15]). Examples of natural bundles are the *tangent bundle* TM as well as any *tensor bundle over M* (order $s = 1$), or the *bundle of linear connections over M* (order $s = 2$). In particular, since we are interested in applications to General Relativity, the bundle $\mathcal{C} = \text{Met}(M, \eta)$ of all metrics of signature $\eta = (p, q)$ is natural; for $\eta = (n, 0)$ we recover the Euclidean case, while for $\eta = (n - 1, 1)$ we recover the Lorentzian case ($n = \dim(M)$). Being \mathcal{C} a natural bundle, any spacetime vector field ξ canonically induces a vector field $\hat{\xi}$ over \mathcal{C} which is called the *natural lift* of ξ over \mathcal{C} . Clearly, $\hat{\xi}$ can be then prolonged to $J^k \mathcal{C}$ by specializing equation (2.1).

A *Lagrangian of order k* is a fibered morphism $L : J^k \mathcal{C} \rightarrow A_n(M)$ into the bundle $A_n(M)$ of n -forms over M ($n = \dim(M)$). Locally, one has

$$L = \mathcal{L}(x^\mu, y^i, y_\mu^i, \dots, y_{\mu_1 \dots \mu_k}^i) \mathbf{ds}$$

where \mathbf{ds} is the standard local volume of M and \mathcal{L} is called the *Lagrangian density*. One can define the morphism δL representing the *variation of the Lagrangian*, i.e. for any vertical vector field X over \mathcal{C}

$$\begin{aligned} \delta L : J^k \mathcal{C} &\rightarrow V^*(J^k \mathcal{C}) \otimes A_n(M) \\ \langle \delta L \circ j^k \sigma \mid j^k X \rangle &= \left[\frac{d}{ds} (L \circ J^k \Phi_s \circ j^k \sigma) \right]_{s=0} \end{aligned} \quad (2.3)$$

where $J^k \Phi_s$ is the (vertical) flow of $j^k X$. We recall that $V^*(J^k \mathcal{C})$ is the dual of the *vertical bundle* $V(J^k \mathcal{C})$ where $j^k X$ lives, so that $\langle \delta L \mid j^k X \rangle$

is a fibered morphism from $J^k\mathcal{C}$ into $A_n(M)$. Once it is computed along the prolongation $j^k\sigma$ of any section σ it produces an n -form over M . The standard *first variation theorem* ensures the existence of a unique *Euler-Lagrange morphism* $\mathbb{E}(L) : J^{2k}\mathcal{C} \rightarrow V^*(\mathcal{C}) \otimes A_n(M)$ and a family of *Poincaré-Cartan morphisms* $\mathbb{F}(L, \Gamma) : J^{2k-1}\mathcal{C} \rightarrow V^*(J^{k-1}\mathcal{C}) \otimes A_{n-1}(M)$ satisfying the *first variation formula*

$$\langle \delta L \mid j^k X \rangle = \langle \mathbb{E}(L) \mid X \rangle + \text{Div} \langle \mathbb{F}(L, \Gamma) \mid j^{k-1} X \rangle \quad (2.4)$$

where Div denotes the *formal divergence operator* which acts on morphisms $\omega : J^h\mathcal{C} \rightarrow A_l(M)$ producing the morphisms $\text{Div} \omega : J^{h+1}\mathcal{C} \rightarrow A_{l+1}(M)$, defined by the equation $(\text{Div} \omega) \circ j^{h+1}\sigma = d(\omega \circ j^h\sigma)$. The Poincaré-Cartan morphism is parametrized by a fibered connection Γ (see [16], [17]); in applications to General Relativity it is standard to take Γ to be the Levi-Civita connection of the dynamical metric.

The Lagrangian L is *natural* if it is generally covariant with respect to the action of $\text{Diff}(M)$ over \mathcal{C} . Infinitesimally, for any spacetime vector field ξ and any section σ of \mathcal{C} , the following identity must hold:

$$\langle \delta L \circ j^k\sigma \mid j^k \mathcal{L}_\xi \sigma \rangle = \mathcal{L}_\xi(L \circ j^k\sigma) \quad (2.5)$$

where

$$\mathcal{L}_\xi \sigma = T\sigma(\xi) - \hat{\xi} \circ \sigma \quad (2.6)$$

is the *Lie derivative* of the section σ .

For standard General Relativity one can choose the Hilbert-Einstein Lagrangian

$$L_{HE} = \frac{1}{16\pi} \sqrt{g} g^{\mu\nu} R_{\mu\nu} \, ds \quad (2.7)$$

where $R_{\mu\nu}$ denotes the Ricci tensor of the covariant metric $g^{\mu\nu}$ and \sqrt{g} denotes the square root of the absolute value of the determinant of the metric. This is a natural Lagrangian.

Because of the covariance condition (2.5) and the first variation formula (2.4), for any spacetime vector field $\xi = \xi^\mu \partial_\mu$ we can define the *Nöther current* and the *work current* given respectively by:

$$\begin{aligned} \mathcal{E}(L, \xi) &= \langle \mathbb{F}(L, \Gamma) \mid j^{k-1} \mathcal{L}_\xi \sigma \rangle - i_\xi L \\ \mathcal{W}(L, \xi) &= - \langle \mathbb{E}(L) \mid \mathcal{L}_\xi \sigma \rangle \end{aligned} \quad (2.8)$$

Both the Nöther current $\mathcal{E}(L, \xi) : J^{2k-1}\mathcal{C} \rightarrow A_{n-1}(M)$ and the work current $\mathcal{W}(L, \xi) : J^{2k}\mathcal{C} \rightarrow A_n(M)$ are well-defined global bundle morphisms (provided

$\mathcal{L}_\xi \sigma$ is interpreted as the *formal Lie derivative*, i.e. the bundle morphism $J^1\mathcal{C} \rightarrow V(\mathcal{C})$ which reduces to the Lie derivative of a section (2.6) when evaluated along σ). From (2.5) and (2.4) the following conservation law holds

$$\text{Div } \mathcal{E}(L, \xi) = \mathcal{W}(L, \xi) \quad (2.9)$$

which expresses *Nöther theorem*. Notice that in this way Nöther theorem is expressed in terms of bundle quantities; one can pull-back all currents along a section σ of \mathcal{C} obtaining the Nöther theorem as a relation between forms over M

$$\begin{aligned} \mathcal{E}(L, \xi, \sigma) &= \mathcal{E}(L, \xi) \circ j^{2k-1}\sigma \\ \mathcal{W}(L, \xi, \sigma) &= \mathcal{W}(L, \xi) \circ j^{2k}\sigma \end{aligned} \quad d \mathcal{E}(L, \xi, \sigma) = \mathcal{W}(L, \xi, \sigma) \quad (2.10)$$

Whenever σ is a solution of field equations, the work $\mathcal{W}(L, \xi, \sigma)$ vanishes so that $\mathcal{E}(L, \xi, \sigma)$ is a closed form on M , i.e. $\mathcal{E}(L, \xi)$ is conserved *on-shell*.

If \mathcal{C} is a natural bundle of order s , i.e. it is associated to $L^s(M)$, the current $\mathcal{E}(L, \xi)$ (as well as $\mathcal{W}(L, \xi)$) is a linear combination of ξ^ν and its symmetrized covariant derivatives (with respect to Γ) up to the (finite) order $k + s - 1$ (s , respectively). Integrating covariantly by parts one can *algorithmically* recast the currents in the following form

$$\begin{aligned} \mathcal{E}(L, \xi) &= \tilde{\mathcal{E}}(L, \xi) + \text{Div } \mathcal{U}(L, \xi) \\ \mathcal{W}(L, \xi) &= \mathcal{B}(L, \xi) + \text{Div } \tilde{\mathcal{E}}(L, \xi) \end{aligned} \quad (2.11)$$

The current $\mathcal{B}(L, \xi)$ turns out to be identically zero (which is known as *the generalized Bianchi identity*), while $\tilde{\mathcal{E}}(L, \xi)$ is called the *reduced current* (which vanishes when pulled-back along a solution σ of field equations) and $\mathcal{U}(L, \xi)$ is called the *superpotential*. Thence in any natural theory (as well as in the larger class of *gauge-natural theories*; see [15], [7]) the Nöther currents $\mathcal{E}(L, \xi)$ are always exact *on-shell*, regardless of the topology of spacetime M (see [18]).

Then a conserved quantity associated to ξ and along a solution σ is defined in a region $D \subset M$ as

$$Q_D(L, \xi, \sigma) = \int_D \mathcal{E}(L, \xi, \sigma) = \int_{\partial D} \mathcal{U}(L, \xi, \sigma) \quad (2.12)$$

In applications to General Relativity only *conserved quantities relative to some background \bar{g}* are in general defined (see [1], [7], [10] and references quoted

therein). For the metric field g and the background \bar{g} one can define the following *covariant* (i.e. natural) Lagrangian

$$\begin{aligned} L_{\text{Tot}} &= L_g + L_{g\bar{g}} + L_{\bar{g}} = \\ &= \frac{1}{16\pi} \left[\sqrt{g} g^{\mu\nu} R_{\mu\nu} - d_\lambda (\sqrt{g} g^{\mu\nu} w_{\mu\nu}^\lambda) - \sqrt{\bar{g}} \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} \right] \mathbf{ds} \end{aligned} \quad (2.13)$$

where we set $R_{\mu\nu}$ and $\bar{R}_{\mu\nu}$ for the Ricci tensor of the metric g and of the background \bar{g} , respectively, and

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda & \text{ Levi-Civita connection of } g & u_{\mu\nu}^\lambda &= \Gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \Gamma_{\nu)}^\alpha{}_\alpha \\ \bar{\Gamma}_{\mu\nu}^\lambda & \text{ Levi-Civita connection of } \bar{g} & \bar{u}_{\mu\nu}^\lambda &= \bar{\Gamma}_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \bar{\Gamma}_{\nu)}^\alpha{}_\alpha \\ & & w_{\mu\nu}^\lambda &= u_{\mu\nu}^\lambda - \bar{u}_{\mu\nu}^\lambda \end{aligned} \quad (2.14)$$

By defining the tensor quantities $q_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \bar{\Gamma}_{\mu\nu}^\lambda$, the Lagrangian (2.13) can be recasted as

$$L_{\text{Tot}} = \frac{1}{16\pi} \left[\sqrt{g} g^{\mu\nu} (q_{\mu\sigma}^\rho q_{\rho\nu}^\sigma - q_{\sigma\rho}^\sigma q_{\mu\nu}^\rho) + (\sqrt{g} g^{\mu\nu} - \sqrt{\bar{g}} \bar{g}^{\mu\nu}) \bar{R}_{\mu\nu} \right] \mathbf{ds} \quad (2.15)$$

showing that it is first order in g and second order in \bar{g} . We also remark that the Lagrangian L_{Tot} vanishes when computed on the background $g = \bar{g}$. A forthcoming paper (see [12]) will be devoted to a detailed discussion about the choice of the Lagrangian (2.13) and the comparison with other choices of the action functional suited to deal with conserved quantities relative to a background (see also [10], [19],[20] and references quoted therein).

The field equations induced by the Lagrangian (2.13) are vacuum Einstein equations for g and \bar{g} , respectively. The Poincaré-Cartan morphism is defined by

$$\begin{aligned} < \mathbb{F}(L_{\text{Tot}}) | j^1 \mathcal{L}_\xi \sigma > = < \mathbb{F}(L_g) | j^1 \mathcal{L}_\xi g > + < \mathbb{F}(L_{g\bar{g}}) | j^1 \mathcal{L}_\xi \sigma > + \\ & + < \mathbb{F}(L_{\bar{g}}) | j^1 \mathcal{L}_\xi \bar{g} > = \\ & = \frac{1}{16\pi} \left[\sqrt{g} (g^{\lambda\alpha} g_{\mu\nu} - \delta_{(\mu}^\lambda \delta_{\nu)}^\alpha) \nabla_\alpha (\mathcal{L}_\xi g^{\mu\nu}) - \mathcal{L}_\xi (\sqrt{g} g^{\mu\nu} w_{\mu\nu}^\lambda) + \right. \\ & \quad \left. - \sqrt{\bar{g}} (\bar{g}^{\lambda\alpha} \bar{g}_{\mu\nu} - \delta_{(\mu}^\lambda \delta_{\nu)}^\alpha) \bar{\nabla}_\alpha (\mathcal{L}_\xi \bar{g}^{\mu\nu}) \right] \mathbf{ds}_\lambda \end{aligned} \quad (2.16)$$

where σ denotes here the pair (g, \bar{g}) while ∇_α and $\bar{\nabla}_\alpha$ denote the covariant derivatives induced by $\Gamma_{\beta\mu}^\alpha$ and $\bar{\Gamma}_{\beta\mu}^\alpha$, respectively.

The Lagrangian (2.13) is natural (when regarded as a Lagrangian for the field pair (g, \bar{g})) and thence allows a superpotential

$$\begin{aligned}\mathcal{U}(L_{\text{Tot}}, \xi) &= \mathcal{U}(L_g, \xi) + \mathcal{U}(L_{g\bar{g}}, \xi) + \mathcal{U}(L_{\bar{g}}, \xi) = \\ &= \frac{1}{16\pi} \left[\sqrt{g} \nabla^\beta \xi^\alpha + \sqrt{g} g^{\mu\nu} w_{\mu\nu}^\beta \xi^\alpha - \sqrt{\bar{g}} \bar{\nabla}^\beta \xi^\alpha \right] \mathbf{ds}_{\alpha\beta}\end{aligned}\quad (2.17)$$

If one considers an ADM foliation of spacetime M by means of “*spacelike*” hypersurfaces $\Sigma_\tau = \{\tau = \text{const}\}$ and denotes by ∞ the spatial infinity of each leaf Σ_τ , the *total conserved quantity of g relative to \bar{g}* with respect to a spacetime vector field ξ is defined according to (2.12) by

$$Q_{\text{Tot}} = \int_{\infty} \mathcal{U}(L_{\text{Tot}}, \xi) \circ j^1 \sigma \quad (2.18)$$

The variation of the conserved quantity Q_{Tot} along a (vertical) vector field X is given by

$$\begin{aligned}\delta Q_{\text{Tot}} &= \int_{\infty} \left(\delta \mathcal{U}(L_g, \xi) - i_\xi \langle \mathbb{F}(L_g) | j^1 X \rangle \right) \circ j^1 g + \\ &+ \int_{\infty} \left(\delta \mathcal{U}(L_{\bar{g}}, \xi) - i_\xi \langle \mathbb{F}(L_{\bar{g}}) | j^1 X \rangle \right) \circ j^1 \bar{g}\end{aligned}\quad (2.19)$$

provided the background \bar{g} is chosen to approach the metric g at infinity (a choice which of course selects a class of suitable backgrounds) so that the following holds true

$$\delta \mathcal{U}(L_{g\bar{g}}, \xi) \Big|_{\infty} = -i_\xi \langle \mathbb{F}(L_g) | j^1 X \rangle \Big|_{\infty} - i_\xi \langle \mathbb{F}(L_{\bar{g}}) | j^1 X \rangle \Big|_{\infty} \quad (2.20)$$

The integral of the quantity $\delta \mathcal{U}(L_{\text{Tot}}, \xi) - i_\xi \langle \mathbb{F}(L_{\text{Tot}}) | j^1 X \rangle$ equals δQ_{Tot} as given by (2.19), since for any Lagrangian L the quantity $\delta \mathcal{U}(L, \xi) - i_\xi \langle \mathbb{F}(L) | j^{k-1} X \rangle$ is independent on the addition of pure divergence terms to the Lagrangian. Thence it is a good candidate for *the density of variation of conserved quantities*. Notice also that “absolute conserved quantities” may (and in some examples they in fact do) diverge, so that only *relative* quantities are defined in general (as we already mentioned above).

This geometric, global and variational framework for conserved quantities is particularly suited to deal with black hole entropy. Entropy is a state function S which satisfies the (generalised) first principle of thermodynamics

$$\delta m = T \delta S + \Omega \delta j \quad (2.21)$$

where m is the *total mass*, j is the *total angular momentum*, Ω is the *angular velocity of the black hole horizon* and T is the *temperature* of the Hawking radiation irradiated by the black hole. Further work terms may occur in general on the r.h.s. of equation (2.21) due to gauge symmetries; they are not considered here since we are going to apply this framework to a natural theory (see [7]).

Inverting (2.21) one immediately obtains for the variation of the entropy

$$\delta S = \frac{1}{T} (\delta m - \Omega \delta j) \quad (2.22)$$

The quantity $(\delta m - \Omega \delta j)$ on the r.h.s. is readily recognised as the Nöther conserved quantity associated to $\xi = \partial_\tau + \Omega \partial_\phi$, so that equation (2.22) can be recasted as

$$\delta S = \frac{1}{T} \int_\infty \left(\delta \mathcal{U}(L_{\text{Tot}}, \xi) - i_\xi \langle \mathbb{F}(L_{\text{Tot}}) | j^1 X \rangle \right) \circ j^1 \sigma \quad (2.23)$$

Quantities on the r.h.s. of this latter equation are known; equation (2.23) can thence be regarded as a variational equation for S that, if integrable, can be integrated to provide at once the entropy and the “*first principle of thermodynamics*” (2.21) it obeys by its own definition.

The following theorem holds in any natural (as well as gauge-natural, see [7]) theory.

For any natural (as well as gauge-natural) Lagrangian L let σ be a solution of field equations, X a solution of linearized field equations (i.e. it drags solutions into solutions) and ξ s a Killing vector for σ (i.e. $\mathcal{L}_\xi \sigma = 0$). Then the form $(\delta \mathcal{U}(L, \xi) - i_\xi \langle \mathbb{F}(L) | j^{k-1} X \rangle) \circ j^{2k-1} \sigma$ is closed.

Specializing this theorem to both terms in the r.h.s. of (2.19), we see that the integral in (2.23) does not depend on the integration region ∞ but just on its homology. Let then Σ be any $(n-2)$ -surface such that $\infty - \Sigma$ is a homological boundary; we can express the variation of entropy as

$$\delta S = \frac{1}{T} \int_\Sigma \left(\delta \mathcal{U}(L_{\text{Tot}}, \xi) - i_\xi \langle \mathbb{F}(L_{\text{Tot}}) | j^{k-1} X \rangle \right) \circ j^1 \sigma \quad (2.24)$$

When dealing with black hole solutions this formula reproduces the “one quarter area law” (see [21]).

We stress that in the definition (2.24) entropy is not *a priori* related to spatial infinity (as conserved quantities are), nor to the horizon of black hole (as sometimes it is asserted in the literature). Due to this fact we may apply formula (2.24) also to solutions other than black hole ones.

3. Taub-Bolt Solution

We are going to describe Taub-Bolt (and Taub-NUT) solutions of vacuum Einstein equations in four-dimensional Euclidean General Relativity . We also give here a short summary of the results obtained in [1], [2] and [3], in order to compare them with our computations in the next Section.

The Taub-Bolt metric

$$\begin{cases} g_{TB} = V_{TB} (d\tau + 2N \cos \theta d\phi)^2 + \frac{dr^2}{V_{TB}} + (r^2 - N^2)(d\theta^2 + \sin^2 \theta d\phi^2) \\ V_{TB} = \frac{(r - 2N)(r - N/2)}{r^2 - N^2} \quad N \text{ constant} \end{cases} \quad (3.1)$$

is regular if we assume $r > 2N$ and we fix the period β of the Euclidean time τ equal to $8\pi N$. This vacuum solution is not asymptotically flat but rather *asymptotically locally flat* (ALF). The boundary at large r is a S^1 bundle with constant circumference β over a sphere S^2 parametrized by θ and ϕ . The fixed points set of the $U(1)$ isometry τ is the two dimensional surface $r = 2N$ parametrized by θ and ϕ . It is called a *bolt* and its area is equal to $12\pi N^2$. Moreover, the metric (3.1) has NUT charge due to the presence of a Misner string (which is a two dimensional coordinate singularity) running along the z -axis from the bolt out to infinity (see [4]).

The first implication of the asymptotic behaviour of Taub-Bolt metric is that it cannot be matched at infinity to flat space. In order to deal with non-compact spacetimes one nevertheless needs a background solution with the same asymptotic behaviour of the solution itself. In Taub-Bolt metric the background is chosen to be the self-dual Taub-NUT solution (see [1]). The general Taub-NUT solution is:

$$\begin{cases} g_{TN} = V_{TN} (d\tau + 2N \cos \theta d\phi)^2 + \frac{dr^2}{V_{TN}} + (r^2 - N^2)(d\theta^2 + \sin^2 \theta d\phi^2) \\ V_{TN} = \frac{r - N}{r + N} \end{cases} \quad (3.2)$$

It is regular if we assume $r > N$ and it has the same ALF behaviour of Taub-Bolt metric. The fixed points set of the $U(1)$ isometry is now described by the zeros of V_{TN} . In this case however, the set $r = N$, called the *nut*, is 0 dimensional because it occurs precisely when the 2-sphere (parametrized by (θ, ϕ)) degenerates. Again, there is a Misner string which now runs along the z axis from the nut out to infinity.

We remark that both Taub-Bolt and Taub-NUT Euclidean solutions are obtained by Wick rotating the same Lorentzian Taub-NUT metric:

$$\begin{cases} g = -f(r) (dt + 2l \cos \theta d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2) \\ f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2} \quad m, l \text{ constants} \end{cases} \quad (3.3)$$

In order to avoid conical singularity at the origin, in the Euclidean sector one has to fix the constant m in terms of $l \rightarrow iN$. The two different fixings:

$$m = \frac{5}{4}N \quad m = N \quad (3.4)$$

give the Taub-Bolt and the Taub-NUT Euclidean solutions, respectively (see [9]).

In order to consider Taub-NUT metric as a well defined background for Taub-Bolt spacetime one requires it to match (at the suitable asymptotic order) the Taub-Bolt solution (3.1) on the “large” sphere of radius ρ (see [1]). To achieve this task, it is necessary to rescale the imaginary time and the nut charge in (3.2). The result is the matched Taub-NUT metric:

$$\begin{cases} \bar{g} = \mu^2 \bar{V} (d\tau + 2N \cos \theta d\phi)^2 + \frac{dr^2}{\bar{V}} + (r^2 - \mu^2 N^2)(d\theta^2 + \sin^2 \theta d\phi^2) \\ \bar{V} = \frac{r - \mu N}{r + \mu N} \quad \mu = \left(1 - \frac{N}{4\rho}\right) \end{cases} \quad (3.5)$$

which agrees with (3.1) at order $O(1/\rho)$ on $r = \rho$.

The starting point for the analysis developed in [1] and [2] is the action functional:

$$I = -\frac{1}{16\pi} \int_D d^4x \sqrt{g} R(g) - \frac{1}{8\pi} \int_{\partial D} d^3x \sqrt{b} [\Theta(b) - \Theta(\bar{b})] \quad (3.6)$$

where D is a compact region of spacetime with regular boundary ∂D , namely the 3-sphere of radius ρ which is let then go to infinity; $b_{\mu\nu}$ is the metric induced on ∂D and $\Theta(b)$ is the trace of the extrinsic curvature of ∂D in D . The bar is again used in order to indicate the objects built out from the background metric.

We recall that the volume integral in (3.6), when evaluated along solutions, gives no contribution to the computation of the action. Moreover Taub-NUT

metric (3.5) matches correctly the Taub-Bolt metric (3.1) on ∂D so that the total action has a finite value:

$$I = \pi N^2 \quad (3.7)$$

We also remark that the action (3.6) is a possible choice for an action suited to describe a manifold with a boundary of fixed intrinsic geometry (additional terms are needed if we are in presence of non orthogonal boundaries, see [20]). Roughly speaking it has a well defined variational principle if subjected to these boundary conditions (i.e. fixed intrinsic three-metric on the boundary). Moreover (3.6) is defined in order to be identically zero when computed along the background solution \bar{g} .

We remark that another choice of a suitable action, instead of (3.6), is the action induced by the Lagrangian (2.13)

$$I_{\text{Tot}} = \int_D L_{\text{Tot}} \circ j^2 \sigma \quad (3.8)$$

Notice that if one computes the value of the total action (3.8) for the Taub-Bolt metric (3.1) with respect to the Taub-NUT matched background (3.5) one obtains $I_{\text{Tot}} = \pi N^2$ as well. Notice also that the same result is obtained using the Taub-NUT solution (3.2) as a background. These results are due to the fact that, fixing ∂D to be the 3-sphere of constant radius $r = \rho$, the action (3.6) agrees with the action (3.8) as ρ approaches infinity (see [12] for greater details).

We also recall (see [1]) the expression of the action in terms of the *Komar mass*:

$$I = \frac{\beta}{2} M_{\text{Kom}} \quad (3.9)$$

which, due to (3.7), gives

$$M_{\text{Kom}} = \frac{N}{4} \quad (3.10)$$

This result, again, must be viewed as relative to the background.

In order to calculate the entropy, a 3+1 decomposition of the action (3.6) was performed in [2]. The main result is that entropy is not just a quarter of the area of the fixed points set of the Euclidean time translation Killing vector field, i.e. of the bolt (as it is for black hole solutions). A further contribution is in fact due to the presence of the Misner string. According to [2] one can associate the entropy of a solution to the obstructions in foliating topologically non trivial Euclidean spacetimes with $\tau = \text{const}$ surfaces which do not intersect and which agree with Euclidean time at infinity. These obstructions arise from the presence of bolts as well as from Misner strings. The $\tau = \text{const}$ surfaces have a boundary at the fixed points set and around Misner string plus the

usual boundary at infinity. Additional boundaries are responsible of further contributions in evaluating the formula for entropy. The final formula obtained in [2] and [3] is

$$S = \frac{1}{4} (A_{\text{Bolt}} + \Delta A_{MS}) - \beta H_{MS} \quad (3.11)$$

where $A_{\text{Bolt}} = 12\pi N^2$ is the area of the bolt, $\Delta A_{MS} = -12\pi N^2$ is the difference between the area of the Misner string in the Taub-Bolt metric minus the same area in the Taub-NUT solution. The term H_{MS} is the Hamiltonian surface term evaluated on the boundary of the Misner string on the $\tau = \text{const}$ surfaces. It is given by the shift of the $3+1$ decomposition of the metric times a component of the second fundamental form of the constant τ surface. The difference of this term from the background is finite and it has been computed to be $H_{MS} = -N/8$ (see [2]). Thus the entropy turns finally out to be:

$$S = \pi N^2 \quad (3.12)$$

4. Conserved Quantities and Entropy

Let us consider the Taub-Bolt metric (3.1) together with the Taub-NUT matched solution (3.5) as background. We shall hereafter specialize to these metrics the formulae introduced in Section 2.

The total conserved quantity (2.18) for the superpotential (2.17) and the vector $\xi = \partial_\tau$, can be readily computed as

$$Q_{\text{Tot}} = \frac{5}{8}N + \frac{N}{8} - \frac{N}{2} = \frac{N}{4} \quad (4.1)$$

which agrees with expression (3.10). We remark that the pure divergence term $L_{g\bar{g}}$ in the Lagrangian (2.13) reverberates in conserved quantities by curing the *anomalous factor* (see [22]); also in this example, in fact, the conserved quantity associated to each Lagrangian L_g and $L_{\bar{g}}$ would equal one half of the expected value.

Analogously, one can apply expression (2.19) to determine the variation of the conserved quantity

$$\delta Q_{\text{Tot}} = \frac{5}{4}\delta N - \delta N = \frac{1}{4}\delta N \quad (4.2)$$

Notice that the contributions due to the Taub-Bolt metric (3.1) and the Taub-NUT matched solution (3.5) can be isolated reproducing the expected values

(see expressions (3.4)). Notice also that pure divergence terms in the Lagrangian, as already remarked, do not contribute to the *variation of conserved quantity*.

According to the previous Section, let us consider $1/T = \beta = 8\pi N$ in expression (2.23); one easily obtains

$$\delta S = \frac{1}{T} \delta Q_{\text{Tot}} = 2\pi N \delta N \quad (4.3)$$

which can be integrated to obtain $S = \pi N^2$. It again agrees with the expected value for the total entropy (3.12).

One can check that the 2-forms $(\delta \mathcal{U}(L, \xi) - i_\xi \langle \mathbb{F}(L) | j^{k-1} X \rangle) \circ j^1 \sigma$ in (2.19) are separately closed for both the Lagrangians $L = L_g$ and $L = L_{\bar{g}}$, as it follows also from the general theory since $\xi = \partial_\tau$ is a Killing vector of both the Taub-Bolt metric and the Taub-NUT background. Then the variation of the entropy can be re-written as an integral over a finite 2-region Σ homologous to ∞ . Of course, because of the Misner string, the region Σ cannot be the Bolt $S_B^2 = \{t = t_0, r = 2N\}$ alone; to obtain a region homologous to ∞ we must also consider two cones $C_B = \{t = t_0, \theta = \theta_0\}$ wrapping around the singularity (due to Misner string) for $\theta_0 \simeq 0$ and $\theta_0 \simeq \pi$. Analogously for the Taub-NUT background, we can consider any region enveloping the nut surface $S_N^2 = \{t = t_0, r \rightarrow \mu N\}$ together with the cones C_N running now from the nut out to infinity. In this way one can compute various contributions to (2.24); letting then the cones close over the z -axis one obtains

$$\begin{aligned} \delta S_B &= \frac{1}{T} \int_{S_B^2} \delta \mathcal{U}(L_g, \xi) - i_\xi \langle \mathbb{F}(L_g) | j^{k-1} X \rangle = 6\pi N \delta N \\ \delta S_N &= \frac{1}{T} \int_{S_N^2} (\delta \mathcal{U}(L_{\bar{g}}, \xi) - i_\xi \langle \mathbb{F}(L_{\bar{g}}) | j^{k-1} X \rangle = -2\pi N \delta N \\ \delta S_{MS} &= \frac{1}{T} \int_{C_B} \delta \mathcal{U}(L_g, \xi) - i_\xi \langle \mathbb{F}(L_g) | j^{k-1} X \rangle = 4\pi N \delta N \\ \delta S_{\overline{MS}} &= \frac{1}{T} \int_{C_N} \delta \mathcal{U}(L_{\bar{g}}, \xi) - i_\xi \langle \mathbb{F}(L_{\bar{g}}) | j^{k-1} X \rangle = 10\pi N \delta N \end{aligned} \quad (4.4)$$

Of course the total variation of the entropy

$$\delta S = \delta S_B + \delta S_{MS} - \delta S_N - \delta S_{\overline{MS}} = 2\pi N \delta N \quad (4.5)$$

agrees with the established result (4.3) and can be integrated to agree again with expression (3.12). We remark that one can isolate the contribution due to

the bolt $S_B = 3\pi N^2$, which equals $\frac{1}{4}A_{\text{Bolt}} = 3\pi N^2$. We also remark that the difference of the contributions along the Misner string $S_{MS} - S_{\overline{MS}} = -3\pi N^2$ equals one quarter of the difference of the areas of the Misner string $\Delta A_{MS} = -3\pi N^2$ with respect of Taub-Bolt and Taub-NUT matched solutions.

5. Conclusions and Perspectives

We reproduced the value of entropy of Taub-Bolt metric *relative* to the Taub-NUT matched background by the geometric framework introduced in Section 2. In that framework entropy is not related to any horizon; in particular we do not need horizons to be Killing so that the Misner string can be treated as well, re-producing the deviation from the one-quarter-area paradigm already noticed by Hawking, Hunter and Page (see [2], [3]). Consequently, our results perfectly agree with the conclusions there achieved: entropy is related to the obstruction to foliate the spacetime with a family of hypersurfaces of constant τ . In fact, the breakdown to foliation comes from coordinate singularities of the metric (fixed points set, as the bolt and nut, as well as the Misner string). These singularities are related to boundary terms in (3.11) and (4.4) which both depend on the 2-homology of hypersurfaces Σ_τ at constant τ .

We finally remark that calculations may be performed also by using the Taub-NUT metric (3.2) as background in place of (3.5). The metric (3.2) agrees with the Taub-Bolt metric (3.1) at infinity as well, but in this case the matching condition required in [1] is not satisfied. In our framework this additional matching condition is not necessary; in fact, all results presented in Section 4 are unaltered by setting (3.2) as a background.

Future investigations will be devoted to the study of the Lorentzian case as well as the asymptotically locally anti-deSitter case (i.e. with cosmological constant). Both these examples appear to fall into our scope since no hypotheses restrict dimension of the spacetime, signature, or asymptotic behaviour of the metric.

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